

HOW RANDOM ARE A LEARNER'S MISTAKES ?

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ABSTRACT. Given a random binary sequence $X^{(n)}$ of random variables, X_t , $t = 1, 2, \dots, n$, for instance, one that is generated by a Markov source (teacher) of order k^* (each state represented by k^* bits). Assume that the probability of the event $X_t = 1$ is constant and denote it by β . Consider a learner which is based on a parametric model, for instance a Markov model of order k , who trains on a sequence $x^{(m)}$ which is randomly drawn by the teacher. Test the learner's performance by giving it a sequence $x^{(n)}$ (generated by the teacher) and check its predictions on every bit of $x^{(n)}$. An error occurs at time t if the learner's prediction Y_t differs from the true bit value X_t . Denote by $\xi^{(n)}$ the sequence of errors where the error bit ξ_t at time t equals 1 or 0 according to whether the event of an error occurs or not, respectively. Consider the subsequence $\xi^{(\nu)}$ of $\xi^{(n)}$ which corresponds to the errors of predicting a 0, i.e., $\xi^{(\nu)}$ consists of the bits of $\xi^{(n)}$ only at times t such that $Y_t = 0$. In this paper we compute an estimate on the deviation of the frequency of 1s of $\xi^{(\nu)}$ from β . The result shows that the level of randomness of $\xi^{(\nu)}$ decreases relative to an increase in the complexity of the learner.

1. INTRODUCTION

Let $X^{(n)} = X_1, \dots, X_n$ be a sequence of binary random variables drawn according to some unknown joint probability distribution $\mathbb{P}(X^{(n)})$. Consider the problem of learning to predict the next bit in a binary sequence drawn according to \mathbb{P} . For training, the learner is given a finite sequence $x^{(m)}$ of bits $x_t \in \{0, 1\}$, $1 \leq t \leq m$, drawn according to \mathbb{P} and estimates a model \mathcal{M} that can be used to predict the next bit of a partially observed sequence. After training, the learner is tested on another sequence $x^{(n)}$ drawn according to the same unknown distribution \mathbb{P} . Using \mathcal{M} he produces the bit y_t as a prediction for x_t , $1 \leq t \leq n$. Denote by $\xi^{(n)}$ the corresponding binary sequence of mistakes where $\xi_t = 1$ if $y_t \neq x_t$ and is 0 otherwise. We pose the following main question: how random is $\xi^{(n)}$?

It is clear that the sequence of mistakes should be random since the test sequence $x^{(n)}$ is random. It may also be that because the learner is using a model of a finite structure (or a finite description-length) that it may somehow introduce dependencies and cause $\xi^{(n)}$ to be less random than $x^{(n)}$. And yet by another intuition, perhaps the fact that the learner is of a finite complexity limits its ability to 'deform' (or distort) randomness of $x^{(n)}$? These are all valid initial guesses that relate to this main question. We note that our basis for saying that \mathcal{M} has a finite structure stems from it being an element of some regular hypothesis class, for instance, having a finite VC-dimension as is often the case in a learning setting (see for instance structural risk minimization of [22]). In the current paper, we are not interested in the learner's performance (as modeled for instance by Valiant's PAC framework [16, 21]) but instead we take a data-centric view and ask how much influence the learner has on the stochastic properties of the errors. We view the learner as an entity that 'interferes' with the randomness that is inherent in the sequence to be predicted and through his predictions creates a sequence of mistakes that has a different stochastic character. This view in a broader sense is taken in

[17] and is shown (empirically) in [18] to explain how static structures may 'deform' random external forces.

To answer the main question raised above we build on a specific learning setting and use it for our analysis. In this setting we consider a teacher that uses a probability distribution \mathbb{P} based on a Markov model with a certain complexity. The learner has access to a hypothesis class of Boolean decision rules that are also based on Markov models. Hence, learning amounts to the estimation of parameters of a finite-order Markov model (see for instance [7, 14]). It is obvious that many different settings can be analyzed, in particular, more general ones. For instance, considering a learner that in addition to parametric estimation, does statistical-model-selection [4].

The remainder of the paper is organized as follows: in section 2 we give a brief introduction to the notion of randomness, in section 3 we define the problem and state our result, and in section 4 we prove the theorem.

2. RANDOMNESS OF A FINITE SEQUENCE

The notion of randomness of finite objects (binary sequences) aims to explain the intuitive idea that a sequence, whether finite or infinite, should be measured as being more unpredictable if it possesses fewer regularities (patterns). There is no formal definition of randomness but there are three main properties that a random binary string of length n must intuitively satisfy [20]. The first property is the so-called *stochasticity* or frequency stability of the sequence which means that any binary word of length $k \leq n$ must have the same frequency limit (equal to 2^{-k}). This is basically the notion of normality that Borel introduced and is related to the degree of unpredictability of the sequence. The second property is *chaoticity* or disorderliness of the sequence. A sequence is less chaotic (less complex) if it has a short description, i.e., if the minimal length of a program that generates the sequence is short. The third property is *typicalness*. A random sequence is a typical representative of the class Ω of all binary sequences. It has no specific features distinguishing it from the rest of the population. An infinite binary sequence is typical if each small subset E of Ω does not contain it (the correct definition of a 'small' set was given by Martin L  f [13]).

As mentioned in section 1, our interest in this paper is essentially to ask what 'interference' does a learner have on the randomness of a test sequence. It appears essential that we look not only on the randomness of the object itself (the test sequence $x^{(n)}$) but also at the interfering entity—the learner, specifically, its algorithmic component that is used for prediction. Related to this, there is an area of research that studies algorithmic randomness which is the relationship between complexity and stochasticity of finite and infinite binary sequences [3]. Algorithmic randomness was first considered by von Mises in 1919 who defined an infinite binary sequence α of zeros and ones as random if it is unbiased, i.e. if the frequency of zeros goes to $1/2$, and every subsequence of α that we can extract using an admissible selection rule (see definition below) is also not biased. Kolmogorov and Loveland [9, 11] proposed a more permissive definition of an admissible selection rule as any (partial) computable process which, having read any n bits of an infinite binary sequence α , picks a bit that has not been read yet, decides whether it should be selected or not, and then reads its value. When subsequences selected by such a selection rule pass the unbiasedness test they are called Kolmogorov-Loveland stochastic (KL-stochastic for short). Martin L  f [13] introduced a notion of randomness which is now considered by many as the most satisfactory notion of algorithmic randomness. His definition says precisely which infinite binary sequences are random and which are not. The definition is probabilistically convincing in

that it requires each random sequence to pass every algorithmically implementable statistical test of randomness.

Let us briefly define what is meant by a selection rule. As mentioned above, this is a principal concept used as part of tests of randomness of sequences. Let $\{0, 1\}^*$ be the space of all finite binary sequences and denote by $\{0, 1\}^n$ the set of all finite binary sequences of length n . An admissible *selection rule* R is defined ([9, 23]) based on three partial recursive functions f, g and h on $\{0, 1\}^*$. Let $x^{(n)} = x_1, \dots, x_n$. The process of selection is recursive. It begins with an empty sequence \emptyset . The function f is responsible for selecting possible candidate bits of $x^{(n)}$ as elements of the subsequence to be formed. The function g examines the value of these bits and decides whether to include them in the subsequence. Thus f does so according to the following definition: $f(\emptyset) = i_1$, and if at the current time k a subsequence has already been selected which consists of elements x_{i_1}, \dots, x_{i_k} then f computes the index of the next element to be examined according to element $f(x_{i_1}, \dots, x_{i_k}) = i$ where $i \notin \{i_1, \dots, i_k\}$, i.e., the next element to be examined must not be one which has already been selected (notice that maybe $i < i_j$, $1 \leq j \leq k$, i.e., the selection rule can go backwards on x). Next, the two-valued function g selects this element x_i to be the next element of the constructed subsequence of x if and only if $g(x_{i_1}, \dots, x_{i_k}) = 1$. The role of the two-valued function h is to decide when this process must be terminated. This subsequence selection process terminates if $h(x_{i_1}, \dots, x_{i_k}) = 1$ or $f(x_{i_1}, \dots, x_{i_k}) > n$. Let $R(x^{(n)})$ denote the selected subsequence. By $K(R|n)$ we mean the length of the shortest program computing the values of f, g and h given n .

From the previous discussion, we know that there are two principal measures related to the information content in a finite sequence $x^{(n)}$, stochasticity (unpredictability) and chaoticity (complexity). An infinitely long binary sequence is regarded random if it satisfies the principle of stability of the frequency of 1s for any of its subsequences that are obtained by an admissible selection rule [9]. Kolmogorov showed that the stochasticity of a finite binary sequence x may be precisely expressed by the deviation of the frequency of ones from some $0 < p < 1$, for any subsequence of $x^{(n)}$ selected by an admissible selection rule R of finite complexity $K(R|n)$. For an object x given another object y he defined in [8] the complexity of x as

$$K(x|y) = \min\{l(\pi) : \phi(\pi, y) = x\} \quad (2.1)$$

where $l(\pi)$ is the length of the sequence π , ϕ is a universal partial recursive function which acts as a description method, i.e., when provided with input (π, y) it gives a specification for x (for more on that see the nice paper by [23]). The chaoticity of $x^{(n)}$ is large if its complexity is close to its length n . The classical work of [1, 2, 8, 23] relates chaoticity to stochasticity. In [1, 2] it is shown that chaoticity implies stochasticity. For a binary sequence s , let us denote by $\|s\|$ the number of 1s in s , then this can be seen from the following relationship (with $p = 1/2$):

$$\left| \frac{\|R(x^{(n)})\|}{l(R(x^{(n)}))} - \frac{1}{2} \right| \leq c \sqrt{\frac{n - K(x^{(n)}|n) + K(R|n) + 2 \log K(R|n)}{l(R(x^{(n)}))}} \quad (2.2)$$

where $l(R(x^{(n)}))$ is the length of the subsequence selected by R and $c > 0$ is some absolute constant. From this it is apparent that as the chaoticity of $x^{(n)}$ grows the stochasticity of the selected subsequence $R(x^{(n)})$ grows (the bias from $1/2$ decreases). Also, the information content of the selection rule, namely $K(R|n)$, has a direct effect on this relationship: the lower $K(R|n)$ the stronger the stability (smaller

deviation of the frequency of 1s from $1/2$). In [5] the other direction which shows that stochasticity implies chaoticity is proved.

So referring back to the initial guesses we made in section 1 concerning our expectation about the randomness of the error sequence $\xi^{(n)}$, we now have a better clue and expect that the more algorithmically complex a learner's prediction rule is the higher the possibility that it distorts (introduces bias into) the randomness of the test sequence $x^{(n)}$. As will be shown, rather than resorting to algorithmic randomness theory (which requires dealing with the non-practical and hard to analyze notion of Kolmogorov complexity) a direct combinatorial approach will do.

3. PROBLEM DEFINITION

Let us denote by $\{0, 1\}^*$ the space of all finite binary sequences. The learning problem consists of predicting the next bit value in a sequence $X^{(n)} = X_1, X_2, \dots, X_n$ of binary random variables drawn randomly according to a probability distribution \mathbb{P} which is defined based on a Markov chain with a finite number of states s . For convenience, we let the state space be the set of natural numbers between 0 and $2^k - 1$ and represent each state $s \in \mathbb{S}_k \equiv \{0, 1, \dots, 2^k - 1\}$ by its unique binary vector $b = [b(1), b(2), \dots, b(k)] \in \{0, 1\}^k$. We alternatively refer to states either by their decimal number s or their binary vector b .

Associated with these states is the transition matrix T where the i^{th} row represents the conditional probability distribution given state i . Consider drawing a random sequence $X^{(n)}$ using the chain by repeatedly making a transition from the current state S_t at time t to the next state S_{t+1} as dictated by T_k . Suppose that $S_t = i$ and $S_{t+1} = j$ then the teacher emits for X_{t+1} the bit value that is appended to b_t in order to obtain b_{t+1} , i.e., the value X_{t+1} satisfies

$$b_j = [b_i(2), b_i(3), \dots, b_i(k), X_{t+1}]$$

where b_j and b_i are the binary vectors corresponding to the states j and i , respectively. Clearly, the structure of the Markov model allows only two outgoing transitions from any given state since X_{t+1} is binary; we call them a *type-1* and *type-0* transitions. Let us denote by \mathcal{M}_k a Markov model (chain) based on transition matrix T_k . We use k^* to denote the order of the teacher's Markov chain (on which \mathbb{P} is based). For any binary sequence $x^{(k+n)}$ of length at least $k > 0$ if we let $b_k \equiv [b_k(1), \dots, b_k(k)] = [x_1, \dots, x_k]$ and define recursively the value of

$$b_t = [b_{t-1}(2), \dots, b_{t-1}(k), x_t] \quad (3.1)$$

for all times $t > k$, where $x_t \in \{0, 1\}$ then the probability of $x^{(k+n)}$ with respect to the teacher's model is defined by

$$\begin{aligned} \mathbb{P}(x^{(k+n)}) &= \mathbb{P}(S_1 = b_k) \mathbb{P}(S_2 = b_{1+k} | S_1 = b_k) \\ &\quad \dots \mathbb{P}(S_{n+1} = b_{n+k} | S_n = b_{n-1+k}). \end{aligned} \quad (3.2)$$

Henceforth, all random binary sequences are assumed to be drawn according to this probability distribution \mathbb{P} which is based on model \mathcal{M}_{k^*} . Neither the value k^* nor the parameters of \mathcal{M}_{k^*} are known to the learner. From basic theory on finite Markov chains, since the matrix T_{k^*} is stochastic (i.e., the sum of the elements in any row equals 1) then \mathcal{M}_{k^*} has a stationary probability distribution (not necessarily unique), which we denote by \mathbb{P}^* . Let us denote by

$$\beta = \mathbb{P}^*(X_t = 1) \quad (3.3)$$

at any time t .

We henceforth assume that the teacher reached stationarity and is producing sequences (both training and testing) with respect to \mathbb{P}^* (for clearer notation we will drop the star and simply write \mathbb{P}).

As a learner, we consider an algorithm that assumes a Markov model \mathcal{M}_k of dimension $k \geq 1$. The learner estimates the probability parameters

$$p_{ij} \equiv \mathbb{P}(S_{t+1} = j | S_t = i), \quad i, j \in \mathbb{S}_k$$

by

$$\hat{p}_{ij} = \frac{\#\{S_{t+1} = j, S_t = i\}}{\#\{S_t = i\}}$$

where $\#\{S_t = i\}$ denotes the number of times state i appeared in the training sequence $x^{(k+m)}$ which drawn randomly by the teacher according to \mathbb{P} and so \hat{p}_{ij} are the frequency of transitions. The first k bits of $x^{(k+m)}$ indicate the initial state of the learner's model as it reads the training sequence, m is the number of state transitions taken by the teacher's model to generate the sequence. Note that p_{ij} are unknowns since they represent the probability of transition from state i to state j in the learner's model \mathcal{M}_k given a random sequence generated according to the teacher probability distribution \mathbb{P} (which is based on the unknown model \mathcal{M}_{k^*}).

After training, the learner is tested on a random test sequence $X^{(k+n)}$ obtained from the teacher based on \mathcal{M}_{k^*} . The learner is repeatedly asked to predict the next bit for each of the last n random bits X_{k+1}, \dots, X_{k+n} , where as above, the first k bits of $X^{(k+n)}$ indicate the starting state of the learner's model as it reads the test sequence. The learner computes the posterior probability $P(X_{t+1} = 1 | S_t)$, based on the learnt model \mathcal{M}_k , which is the probability that the next bit $X_{t+1} = 1$ given that the current state is S_t (at any given time t the current state consists of the last k bits seen in the test sequence up to t). The learner's decision (prediction) is based on the maximum *a posteriori* probability which is defined as follows: suppose that the current state is i then the decision is

$$d(i) \equiv \begin{cases} 1 & \text{if } \hat{p}(1|i) > 1 - \hat{p}(1|i) \\ 0 & \text{otherwise,} \end{cases} \quad (3.4)$$

$i \in \mathbb{S}_k$, where $\hat{p}(1|i)$ is defined as \hat{p}_{ij} for the state j whose $b_j = [b_i(2), \dots, b_i(k), 1]$ (a type-1 transition from state i) and the corresponding true probability (measured according to \mathbb{P}) is denoted by $p(1|i) = p_{ij}$. Note that (3.4) may be expressed alternatively as

$$d(i) = \begin{cases} 1 & \text{if } \hat{p}(1|i) > \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases} \quad (3.5)$$

Denote by $m_i \geq 0$ the number of times state i was entered as the teacher scans the training sequence $x^{(k+m)}$ from $t = k + 1$ up to $t = k + m$ (as mentioned above, the initial state at time $t = k + 1$ is the state whose $b = [x_1, x_2, \dots, x_k]$). We will sometimes refer to m_i as the i^{th} subsample size. Note that m_i , $i \in \mathbb{S}_k$, are dependent random variables since the Markov chain may visit each state a random number of times and they all must satisfy $\sum_{i=0}^{2^k-1} m_i = m$. We claim that the $\hat{p}(1|i)$, $i \in \mathbb{S}_k$, are independent random variables when conditioned on the vector $[m_0, \dots, m_{2^k-1}]$ (which we henceforth denote by \underline{m}). To see this, consider a training sequence $x^{(m+k)}$ generated by the teacher according to (3.2). Let us denote the corresponding sequence of states by $\sigma^{(m)} = (\sigma_1, \dots, \sigma_m)$ with $\sigma_i \in \mathbb{S}_k$. To show the dependence of $x^{(k+m)}$ on $\sigma^{(m)}$ we will sometimes write $x^{(m+k)} = x(\sigma^{(m)})$. Then

by (3.2) we have

$$\begin{aligned} \mathbb{P}\left(x^{(m+k)}\right) &= \mathbb{P}\left(\sigma^{(m)}\right) \\ &= \mathbb{P}\left(S_1 = \sigma_1\right) \mathbb{P}\left(S_2 = \sigma_2 \mid S_1 = \sigma_1\right) \mathbb{P}\left(S_3 = \sigma_3 \mid S_2 = \sigma_2\right) \\ &\quad \cdots \mathbb{P}\left(S_m = \sigma_m \mid S_{m-1} = \sigma_{m-1}\right) \end{aligned}$$

Due to the structure of the Markov chain not every sequence $\sigma^{(m)} \in \mathbb{S}_k^m$ is possible. Denote by $V \subset \mathbb{S}_k^m$ the set of valid state sequences $\sigma^{(m)}$ (possible under \mathbb{P}). We now show that given that the state sequence $\sigma^{(m)}$ corresponding to the data $x^{(m+k)}$ is in V and conditioned on \underline{m} then any other state sequence that visits the same states the same number of times as $\sigma^{(m)}$ (perhaps in a different order) must have the same probability according to \mathbb{P} . For any $i \in \mathbb{S}_k$ denote by $N_\sigma(1|i)$ the number of type-1 transitions from state i in the sequence σ . Without loss of generality, let us assume that always initially the state is $i = 0$ so that we can write for the first factor $\mathbb{P}\left(S_1 = \sigma_1 \mid 0\right)$. Since all state transitions are either type-0 or type-1 then we have

$$\mathbb{P}\left(x\left(\sigma^{(m)}\right) \mid \underline{m}, \sigma^{(m)} \in V\right) = \prod_{i \in \mathbb{S}_k} (p(1|i))^{N_\sigma(1|i)} (1 - p(1|i))^{m_i - N_\sigma(1|i)} \quad (3.6)$$

where $p(1|i)$ was defined above. Let α be a non-negative integer constant and define the vector function $N(i) \equiv [N(1|i), \alpha - N(1|i)]$. Associate a conditional probability function for the random variable $N(i)$ as

$$\mathbb{P}\left(N(i) = [\ell, \alpha - \ell] \mid \alpha\right) = (p(1|i))^\ell (1 - p(1|i))^{\alpha - \ell}.$$

Then (3.6) may be written as

$$\mathbb{P}\left(x^{(m+k)} \mid \underline{m}, \sigma^{(m)} \in V\right) = \prod_{i \in \mathbb{S}_k} \mathbb{P}\left(N(i) = [N_\sigma(1|i), m_i - N_\sigma(1|i)]\right). \quad (3.7)$$

For a fixed value of m_i the event " $N(i) = [N_\sigma(1|i), m_i - N_\sigma(1|i)]$ " is equivalent to the event " $\hat{p}(1|i) = \frac{N_\sigma(1|i)}{m_i}$ ". Hence alternatively (3.7) can be expressed as

$$\mathbb{P}\left(x^{(m+k)} \mid \underline{m}, \sigma^{(m)} \in V\right) = \prod_{i \in \mathbb{S}_k} \mathbb{P}\left(\hat{p}(1|i) = \frac{N_\sigma(1|i)}{m_i}\right). \quad (3.8)$$

The right side of (3.8) is a product of probability functions of the random variables $\hat{p}(1|i)$. So conditioned on \underline{m} and on the event that $x^{(m+k)}$ corresponds to a valid state sequence $\sigma^{(m)}$ then the event that $x^{(m+k)}$ is generated by the teacher is equivalent to the event that $\sigma^{(m)}$ has transition frequencies $\hat{p}(1|i)$ that independently take the particular values $N_\sigma(1|i)/m_i$ according to $x^{(m+k)}$. It also follows that $\hat{p}(1|i)$ is the average of independent Bernoulli trials (success taken as a type-1 transition from state i) hence is distributed according to the Binomial distribution with parameters m_i and $p(1|i)$.

Let us state the main result of this paper.

Theorem 1. *Let $0 < \delta < 1$ and k, ℓ, m, n be positive integers. Let \mathbb{P} be the stationary probability distribution based on a finite, ergodic and reversible Markov chain with probability-transition matrix T that has a second largest eigenvalue λ . Denote by $\gamma = (1 - \max\{0, \lambda\}) / (1 + \max\{0, \lambda\})$. Suppose after reaching stationarity the teacher generates a binary sequence $X^{(n)} = X_1, X_2, \dots, X_n$ by repeatedly drawing X_t according to \mathbb{P} and denote by $\beta = \mathbb{P}(X_t = 1)$. Let $x^{(k+m)}$ be a given*

randomly drawn training sequence according to \mathbb{P} . Suppose that a learner uses $x^{(k+m)}$ to estimate the values of the parameters of a Markov model \mathcal{M}_k with states $i \in \mathbb{S}_k = \{0, \dots, 2^k - 1\}$. Denote by $p(1|i)$ the probability of making a transition from state i to a state whose binary vector is $[b_i(2), \dots, b_i(k), 1]$ (according to T). Let ρ_i denote the probability that a Binomial random variable with parameters m_i and $p(1|i)$ is larger than $m_i/2$ and denote by $\rho^{(m)}$ the average $\frac{1}{2^k} \sum_{i \in \mathbb{S}_k} \rho_i$ with $m = \sum_{i \in \mathbb{S}_k} m_i$. Suppose that the learner is tested incrementally on a randomly drawn sequence $x^{(k+n)}$ generated according to \mathbb{P} . The learner predicts an output bit y_t for every input bit x_t in $x^{(k+n)}$ using \mathcal{M}_k and the maximum a posteriori decision. Denote by $\xi^{(k+n)}$ the sequence of mistakes where $\xi_t = 1$ if $y_t \neq x_t$, and $\xi_t = 0$ otherwise, $k+1 \leq t \leq k+n$. Suppose that the subsequence $\xi^{(\nu)}$ of mistakes corresponding to 0-valued predictions is of length $\nu \geq \ell$. Denote by $\epsilon(\ell, k, \rho^{(m)}, \delta)$ the following expression

$$\sqrt{\max \left\{ \frac{1}{2\ell\gamma} \left(\ln \left(\frac{4}{\delta} \right) + 3\rho^{(m)}2^{k-1} \left(\ln \left(\frac{1}{\rho^{(m)}} \right) + 1 \right) + k \right), \frac{1}{2^k \rho^{(m)}} \ln \left(\frac{1}{\delta} \right) \right\}}.$$

Then for any $0 < \delta < 1$, with confidence at least $1 - \delta$ the deviation between β and the frequency of 1s of $\xi^{(\nu)}$ is bounded as

$$\left| \frac{1}{\nu} \sum_{j=1}^{\nu} \xi_j^{(\nu)} - \beta \right| \leq \epsilon(\ell, k, \rho^{(m)}, \delta)$$

where it is assumed that $(1 + \epsilon(\ell, k, \rho^{(m)}, \delta))\rho^{(m)} < 1/2$.

Let us interpret this theorem in the context of the discussion in sections 1 and 3. We see that the quantity $\rho^{(m)}2^k$ in the bound plays a role of effective complexity of the learner's decision rule. Let us assume that it is not too low since otherwise the learner's decision rule may be atypical (the number of 1s in the vector d will deviate largely from the mean $2^k \rho^{(m)}$) which is an event of exponentially-small probability (with respect to the learner's model order). Now, under this assumption it is the first expression in the $\max\{\}$ which is important. We see that the larger $\rho^{(m)}2^k$ the more that the mistake sequence $\xi^{(\nu)}$ can deviate in randomness. The theorem therefore implies that a Markov learner who trains on a random binary sequence (generated by a fixed Markov source) will with probability $1 - \delta$ learn a decision rule that makes a sequence of mistakes $\xi^{(\nu)}$, $\nu \geq \ell$, that may deviate from being a purely random sequence (according to \mathbb{P}) by an amount no larger than $\epsilon(\ell, k, \rho, \delta)$. In other words, its frequency of 1s may deviate from the true probability of a 1 by an amount that increases like $O\left(\sqrt{\frac{\rho^{(m)}2^k}{\ell}}\right)$. Another implication of theorem is

that the only explicit dependence on the source comes through γ and $\rho^{(m)}$ via the distribution \mathbb{P} . The value of γ is bounded by 0 and 1 where it is 1 if the Markov chain consists of independent states and gets closer to zero as the states become more mutually dependent. Thus from the bound we see that the more dependent the sequence generated by the source the larger the possible mistake-sequence's deviation in randomness. There is no explicit dependence on k^* however, implicitly, the bound does depend on k^* since when the learner's model-order k is much smaller than the source's order k^* then the 'memory' of the source is larger than the learner's window size (recall that k represents the number of bits per state). As the learner scans the training sequence using a small window (compared to the source's memory) it estimates the state-transition probabilities p_{ij} and obtains $\hat{p}(1|i)$ which on average are close to $1/2$ (and so will the ρ_i). Therefore $\rho^{(m)}$ will be relatively large (recall that we assume that its maximum value is at most $(1 + \epsilon)^{1/2}$). However if $k \geq k^*$ then the learner's window is close to the source's memory

length so in general the \hat{p}_{ij} may deviate considerably from $1/2$. In this case the ρ_i can be close to zero and hence make $\rho^{(m)}$ much smaller than $1/2$. Hence when the learner's model order (compared to the source's order) is small there may be a large deviation of randomness for the mistake sequence. Once the learner's order matches that of the teacher's then the effective complexity may be significantly lower hence making the mistake-sequence's deviation in randomness smaller.

4. PROOF OF THEOREM 1

We assume that the length m of the training sequence $x^{(k+m)}$ (which is drawn randomly according to \mathbb{P}) is fixed. Referring to (3.5), the decision rule of the learner is denoted by a binary vector

$$d = [d(0), \dots, d(2^k - 1)]. \quad (4.1)$$

Note that d fully describes the learner's prediction rule associated with model \mathcal{M}_k (which is learnt based on $x^{(k+m)}$). It describes every possible prediction that can be made for all possible situations (present states). From the previous section, the elements $d(i)$ depend on the random variables $\hat{p}(1|i)$ which when conditioned on \underline{m} are independent and Binomially distributed with parameters m_i and $p(1|i)$, $i \in \mathbb{S}_k$, respectively. We have from (3.5) that

$$\mathbb{P}(d(i) = 1) = \mathbb{P}\left(\hat{p}(1|i) > \frac{1}{2}\right) \equiv \rho_i \quad (4.2)$$

which equals the probability that a Binomial random variable (with parameters m_i , $p(1|i)$) is larger than $m_i/2$. From (4.2) it follows that the elements of d are non-identically distributed Bernoulli random variables with parameters ρ_i .

For a binary vector d denote by $\|d\|$ the l_1 -norm (or Hamming weight) of d . By the statement of the theorem, letting $m = \sum_{i \in \mathbb{S}_k} m_i$ we have

$$\rho^{(m)} \equiv \left(\frac{1}{2^k} \sum_{i=0}^{2^k-1} \rho_i \right) \quad (4.3)$$

then the expected number of 1s in d is

$$\mathbb{E}[\|d\|] = \mathbb{E}\left[\sum_{i=0}^{2^k-1} d(i)\right] = 2^k \rho^{(m)}.$$

Let us define the following set,

$$A_{\epsilon, m}^{(k)} = \left\{ d \in \{0, 1\}^{2^k} : 1 - \epsilon \leq \frac{\|d\|}{2^k \rho^{(m)}} \leq 1 + \epsilon \right\} \quad (4.4)$$

which depends on the subsample size vector $\underline{m} = [m_0, \dots, m_{2^k-1}]$ through $\rho^{(m)}$. Under \mathbb{P} the probability of the event of not falling in $A_{\epsilon, m}^{(k)}$ is the same as the probability of this event conditioned on the state sequence (corresponding to $x^{(k+m)}$) being valid hence we have

$$\begin{aligned} \mathbb{P}\left(d \notin A_{\epsilon, m}^{(k)}\right) &= \mathbb{P}\left(d \notin A_{\epsilon, m}^{(k)} \mid \sigma^{(m)} \in V\right) \\ &= \sum_{\underline{m}} \mathbb{P}\left(d \notin A_{\epsilon, m}^{(k)} \mid \underline{m}, \sigma^{(m)} \in V\right) \mathbb{P}\left(\underline{m} \mid \sigma^{(m)} \in V\right) \end{aligned} \quad (4.5)$$

where the sum ranges over all non-negative \underline{m} that satisfy $\sum_{i \in \mathbb{S}_k} m_i = m$. We now bound the first factor inside the sum by a quantity which only depends on m (not on the specific vector \underline{m}). We have,

$$\begin{aligned}
& \mathbb{P} \left(d \notin A_{\epsilon, m}^{(k)} \middle| \underline{m}, \sigma^{(m)} \in V \right) \\
&= \mathbb{P} \left(\left\{ d : \|d\| > (1 + \epsilon) 2^k \rho^{(m)} \right\} \cup \left\{ d : \|d\| < (1 - \epsilon) 2^k \rho^{(m)} \right\} \middle| \underline{m}, \sigma^{(m)} \in V \right) \\
&\leq \mathbb{P} \left(\|d\| > (1 + \epsilon) 2^k \rho^{(m)} \middle| \underline{m}, \sigma^{(m)} \in V \right) + \mathbb{P} \left(\|d\| < (1 - \epsilon) 2^k \rho^{(m)} \middle| \underline{m}, \sigma^{(m)} \in V \right).
\end{aligned} \tag{4.6}$$

As stated above, conditioned on \underline{m} the $d(i)$ are independent thus $\|d\|$ is a sum of independent non-identically distributed Bernoulli random variables (also known as Poisson trials). We will use the following lemma.

Lemma 2. *Let X_1, \dots, X_n be independent Bernoulli random variables $P(X_i = 1) = p_i$ where $0 \leq p_i \leq 1$ and denote by $p = \frac{1}{n} \sum_{i=1}^n p_i$. Then for any $0 < \epsilon \leq 1$ we have*

$$P \left(\frac{1}{n} \sum_{i=1}^n X_i < (1 - \epsilon)p \right) < e^{-np\epsilon^2/2}$$

and

$$P \left(\frac{1}{n} \sum_{i=1}^n X_i > (1 + \epsilon)p \right) \leq e^{-np\epsilon^2/4}.$$

The slight asymmetry in the bounds can be seen from the proof of the lemma which is based on applying Chernoff bound on the tail probability of the sum of Poisson trials (see Theorem 4.1 and 4.3, in [15]).

By the above lemma and from (4.4), (4.5), (4.6) it follows that the probability that a random d does not fall in $A_{\epsilon, m}^{(k)}$ is bounded as

$$\mathbb{P} \left(d \notin A_{\epsilon, m}^{(k)} \right) \leq 2e^{-2^k \rho^{(m)} \epsilon^2/4}. \tag{4.7}$$

Next, we estimate the cardinality of the set $A_{\epsilon, m}^{(k)}$. From (4.4) we have,

$$|A_{\epsilon, m}^{(k)}| \leq \sum_{i=\lfloor (1-\epsilon)2^k \rho^{(m)} \rfloor}^{\lceil (1+\epsilon)2^k \rho^{(m)} \rceil} \binom{2^k}{i}.$$

Denote by $B(k, n) = \binom{n}{k}$, then it is easy to verify (see for instance [6]) that the ratio

$$\phi(k) = \frac{B(k, n)}{B(k-1, n)} = \frac{n-k+1}{k}$$

decreases monotonically as k increases from 1 to n . For $k > n/2$ this ratio is smaller than 1 hence it follows that for

$$\frac{\binom{n}{k+v}}{\binom{n}{k}} = \phi(k+1)\phi(k+2)\cdots\phi(k+v) \leq \left(\frac{n-k}{k+1} \right)^v.$$

It follows that for any $c > n/2$ if we denote by $\alpha_{c+1} \equiv \phi(c+1) = \frac{n-c}{c+1}$ then the following upper bound holds,

$$\sum_{k=c}^{c+v} \binom{n}{k} \leq \binom{n}{c} (1 + \alpha_{c+1} + \alpha_{c+1}^2 + \cdots + \alpha_{c+1}^v) \leq \binom{n}{c} \frac{1}{1 - \alpha_{c+1}}.$$

Similarly, the inverse $\phi^{-1}(k)$ increases monotonically as k increases and for $k < n/2$ is smaller than 1 hence it follows that for any $c < n/2$ we have

$$\sum_{k=c-v}^c \binom{n}{k} \leq \binom{n}{c} (1 + \alpha_c^{-1} + \alpha_c^{-2} + \dots + \alpha_c^{-v}) \leq \binom{n}{c} \frac{1}{1 - \alpha_c^{-1}}.$$

Therefore, as a bound on the cardinality of $A_{\epsilon, m}^{(k)}$ we have

$$|A_{\epsilon, m}^{(k)}| \leq \begin{cases} \binom{2^k}{\lfloor (1-\epsilon)2^k \rho^{(m)} \rfloor} \frac{1}{1-\alpha_+} & \text{if } \epsilon < 1 - \frac{1}{2\rho^{(m)}} \\ \binom{2^k}{\lceil (1+\epsilon)2^k \rho^{(m)} \rceil} \frac{1}{1-\alpha_-^{-1}} & \text{if } \epsilon < \frac{1}{2\rho^{(m)}} - 1 \end{cases} \quad (4.8)$$

where

$$\alpha_+ = \frac{2^k - \lfloor (1-\epsilon)2^k \rho^{(m)} \rfloor}{\lfloor (1-\epsilon)2^k \rho^{(m)} \rfloor + 1}$$

and

$$\alpha_-^{-1} = \frac{\lceil (1+\epsilon)2^k \rho^{(m)} \rceil}{2^k - \lceil (1+\epsilon)2^k \rho^{(m)} \rceil + 1}.$$

By the assumption of the theorem, ρ_i (defined in (4.2)) have an average value (4.3) that satisfies $(1+\epsilon)\rho^{(m)} < \frac{1}{2}$ so the bottom bound in (4.8) applies. Hence the bound simplifies to

$$|A_{\epsilon, m}^{(k)}| \leq \binom{2^k}{(1+\epsilon)2^k \rho^{(m)}} \frac{1}{1 - \alpha_-^{-1}} \quad (4.9)$$

where we henceforth drop the $\lceil \cdot \rceil$ from the lower entry and leave it implicit.

We now continue the analysis in order to obtain a bound on the possible deviation in randomness of the learner's mistake sequence. Let us denote by $R_d : \{0, 1\}^* \rightarrow \{0, 1\}$ the learner's decision rule which is defined based on the model \mathcal{M}_k learnt by the learner where d is defined in (4.1). When given a finite random binary sequence $X^{(t)}$, R_d produces a binary prediction at time $t+1$, referred to as an *output* bit, which equals

$$\begin{aligned} Y_{t+1} &= R_d(X^{(t)}) \\ &= d(S_t) \end{aligned}$$

where S_t is the state of the learner at time t and $d(S_t)$ is as defined in (3.4). Let us denote by $\xi^{(n)}$ the sequence of errors where the error bit ξ_t at time t equals 1 or 0 according to whether the event of an error in prediction occurs or not, respectively, that is, for a given input sequence $x^{(n)}$ and a prediction sequence $y^{(n)}$ we define

$$\xi_t = \begin{cases} 1 & \text{if } y_t \neq x_t \\ 0 & \text{otherwise.} \end{cases}$$

Consider the subsequence $\xi^{(\nu)}$ of $\xi^{(n)}$ corresponding to the errors associated with the prediction of a 0, i.e., $\xi^{(\nu)}$ consists of the bits of $\xi^{(n)}$ at times t when the prediction bit $y_t = 0$. Clearly, $\xi^{(\nu)}$ is a subsequence of the input $x^{(n)}$ since when the prediction is 0 the error bit equals the input bit. The length ν of this subsequence is a random variable since it depends on the learnt model \mathcal{M}_k . Since $\xi^{(\nu)}$ is a subsequence of the error resulting from prediction by R_d and is also a subsequence of the input $x^{(k+n)}$ we associate a selection rule Γ_d (see section 2) with the decision rule R_d and say that Γ_d selects $\xi^{(\nu)}$ from $x^{(n)}$.

Let $E_{d, \epsilon}^{(\ell)}$ denote the event that based on a given *fixed* rule Γ_d the selected subsequence $\xi^{(\nu)}$ from a random input sequence $x^{(n)}$ is of length at least ℓ and its frequency of 1s deviates from the expected value by at least ϵ , formally,

$$E_{d,\epsilon}^{(\ell)} = \left\{ x^{(n)} : \xi^{(\nu)} = \Gamma_d \left(x^{(n)} \right), \nu \geq \ell, \left| \frac{\|\xi^{(\nu)}\|}{\nu} - \beta \right| > \epsilon \right\},$$

where $\|\xi^{(\nu)}\|$ denotes the number of 1s in the binary sequence $\xi^{(\nu)}$ of length ν .

We use the following lemma which states a rate on the strong law of large numbers for a Markov Chain.

Lemma 3. *Let X_1, \dots, X_n be a finite ergodic and reversible Markov chain in stationary state with a second largest eigenvalue λ and f is a function taking values in $[0, 1]$ such that $\mathbb{E}f(X_i) = \mu$. Denote by $\lambda_0 = \max\{0, \lambda\}$ and the stationary probability distribution \mathbb{P}^* . Then for all $\epsilon > 0$ such that $\mu + \epsilon < 1$ and all time n the following bound holds:*

$$\exp\{-nt(\mu + \epsilon)\} \mathbb{E} \exp \left\{ t \sum_{i=1}^n f(X_i) \right\} \leq e^{-2n\epsilon^2 \left(\frac{1-\lambda_0}{1+\lambda_0} \right)}$$

where the expectation is taken with respect to \mathbb{P}^* .

The lemma follows from the proof of Theorem 1 of [10].

We now apply this lemma. Consider the state subsequence $S^{(\nu)}$ corresponding to the subsequence $\xi^{(\nu)}$ and for any state $s \in \mathbb{S}_k$ let the function $f(s)$ in the lemma take as value the least significant bit of the binary-representation of s so that for the state at time i we have $f(S_i) = \xi_i^{(\nu)}$. Thus in our case μ in the lemma equals $\mathbb{P}(\xi_1^{(\nu)} = 1) = \beta$ since the stationary distribution is invariant for all times and in particular for those times where the prediction is 0. Also, λ is the second largest eigenvalue of the transition matrix T . By the assumption of Theorem 1 the Markov chain of the teacher satisfies the conditions of the lemma. Now, the random subsequence $S^{(\nu)}$ may consist of a mix of random variables S_i that are mutually independent (if their indices are not consecutive) and dependent (otherwise). Suppose that we have a sequence $S^{(\nu,q)} = \{S_i\}_{i=1}^\nu$ with q independent states and denote the corresponding mistake sequence by $\xi^{(\nu,q)}$. Denote by $S^{(q)}$ and $S^{(\nu-q)}$ the parts of $S^{(\nu,q)}$ that are independent and dependent, respectively. That is, $S^{(q)} = \{S_{i_1}, S_{i_2}, \dots, S_{i_q}\}$ are states visited by the subsequence $S^{(\nu)}$ where for any pair $S_{i_j}, S_{i_k} \in S^{(q)}$ the indices are not consecutive $|i_j - i_k| > 1$ hence $S^{(q)}$ is a sequence of independent random states. The other sequence $S^{(\nu-q)} = \{S^{(r_1)}, S^{(r_2)}, \dots, S^{(r_N)}\}$ consists only of consecutive subsequences (referred to as parts) $S^{(r_j)} = \{S_{i_j}, S_{i_j+1}, \dots, S_{i_j+r_j}\}$, $j = 1, \dots, N$, $\sum_{j=1}^N r_j = \nu - q$ where each part consists of consecutive states hence pairwise-dependent. Denote by $\mathcal{I}_q, \mathcal{I}_{\nu-q}, \mathcal{I}_{r_j} \subseteq \mathbb{S}_k$ the sets of time-indices (with respect to the sequence $S^{(\nu)}$) of states in $S^{(q)}$, $S^{(\nu-q)}$ and $S^{(r_j)}$, respectively. We now apply Markov's inequality (see for instance [19]) to the subsequence $\xi^{(\nu,q)}$ and obtain

$$\mathbb{P} \left(\frac{1}{\nu} \sum_{i=1}^\nu \xi_i^{(\nu,q)} - \beta > \epsilon \right) = \mathbb{P} \left(\frac{1}{\nu} \sum_{i=1}^\nu f(S_i) > \beta + \epsilon \right) \leq \exp\{-\nu t(\beta + \epsilon)\} \mathbb{E} \exp \left\{ t \sum_{i=1}^\nu f(S_i) \right\}. \quad (4.10)$$

Due to independence of some of the parts of $S^{(\nu,q)}$ we may split the expectation as follows:

$$\begin{aligned}
\mathbb{E} \exp \left\{ t \sum_{i=1}^{\nu} f(S_i) \right\} &= \mathbb{E} \exp \left\{ t \sum_{i \in \mathcal{I}_q} f(S_i) \right\} \mathbb{E} \exp \left\{ t \sum_{i \in \mathcal{I}_{\nu-q}} f(S_i) \right\} \\
&= \mathbb{E} \exp \left\{ t \sum_{i \in \mathcal{I}_q} f(S_i) \right\} \prod_{j=1}^N \mathbb{E} \exp \left\{ t \sum_{i \in \mathcal{I}_{r_j}} f(S_i) \right\}. \quad (4.11)
\end{aligned}$$

An independent sequence of random variables can be represented as a Markov chain with a row-stochastic transition matrix T all of whose rows are identical. By the Perron-Frobenius theorem such a matrix has a maximum eigenvalue of 1 and from a standard bound on the absolute value of any other eigenvalue (based on Gerschgorin disks [12]) it follows that the second largest eigenvalue equals 0. Thus from Lemma 3 we obtain

$$\exp \{-qt(\beta + \epsilon)\} \mathbb{E} \exp \left\{ t \sum_{i \in \mathcal{I}_q} f(S_i) \right\} \leq \exp \{-2q\epsilon^2\}.$$

Also from the lemma, we get for the inner part of the product in (4.11)

$$\begin{aligned}
&\exp \{-t(\nu - q)(\beta + \epsilon)\} \prod_{j=1}^N \mathbb{E} \exp \left\{ t \sum_{i \in \mathcal{I}_{r_j}} f(S_i) \right\} \\
&\leq \prod_{j=1}^N \exp \left\{ -2r_j \epsilon^2 \left(\frac{1 - \lambda_0}{1 + \lambda_0} \right) \right\} \\
&= \exp \left\{ -2(\nu - q) \epsilon^2 \left(\frac{1 - \lambda_0}{1 + \lambda_0} \right) \right\}
\end{aligned}$$

where again $\lambda_0 = \max \{0, \lambda\}$. Let $\gamma = (1 - \lambda_0)/(1 + \lambda_0)$ then continuing to bound the right hand side of (4.10) we get the following large deviation bound

$$\begin{aligned}
\mathbb{P} \left(\left| \frac{1}{\nu} \sum_{i=1}^{\nu} \xi_i^{(\nu, q)} - \beta \right| > \epsilon \right) &\leq 2 \exp \{-2q\epsilon^2\} \exp \{-2(\nu - q)\gamma\epsilon^2\} \\
&= 2 \exp \{-2q(1 - \gamma)\epsilon^2\} \exp \{-2\nu\gamma\epsilon^2\}.
\end{aligned}$$

It follows that for any fixed $d \in \{0, 1\}^{2^k}$ we have

$$\begin{aligned}
\mathbb{P} \left(E_{d, \epsilon}^{(\ell)} \right) &= \sum_{\nu \geq \ell} \mathbb{P} \left(\left| \frac{\|\xi^{(\nu)}\|}{\nu} - \beta \right| > \epsilon \middle| \nu \right) \mathbb{P}(\nu) \\
&= \sum_{\nu \geq \ell} \sum_{q=0}^{\nu} \mathbb{P} \left(\left| \frac{\|\xi^{(\nu, q)}\|}{\nu} - \beta \right| > \epsilon \middle| \nu, q \right) \mathbb{P}(q|\nu) \mathbb{P}(\nu) \\
&\leq 2 \sum_{\nu \geq \ell} \exp \{-2\nu\gamma\epsilon^2\} \left[\sum_{q=0}^{\nu} \exp \{-2q(1 - \gamma)\epsilon^2\} \mathbb{P}(q|\nu) \right] \mathbb{P}(\nu).
\end{aligned}$$

The inner sum is bounded from above by 1 hence we obtain the simple bound

$$\mathbb{P} \left(E_{d, \epsilon}^{(\ell)} \right) \leq 2 \exp \{-2\ell\gamma\epsilon^2\}. \quad (4.12)$$

Denote by \hat{d} the binary vector associated with the learnt model \mathcal{M}_k (which is based on a random training sequence $x^{(k+m)}$). We are interested in the probability of the event $E_{\hat{d}, \epsilon}^{(\ell)}$ that after learning, given a random test sequence $x^{(n)}$ for prediction, the

learner based on the selection rule $\Gamma_{\hat{d}}$ selects a subsequence $\xi^{(\nu)}$ from $x^{(n)}$ of length at least ℓ which is biased away from β by an amount greater than ϵ .

Denoting by $A_{\epsilon,m}^{(k)c}$ the complement of the set $A_{\epsilon,m}^{(k)}$ then we have

$$\begin{aligned} \mathbb{P}\left(E_{\hat{d},\epsilon}^{(\ell)}\right) &= \mathbb{P}\left(E_{\hat{d},\epsilon}^{(\ell)} \mid \hat{d} \in A_{\epsilon,m}^{(k)}\right) \mathbb{P}\left(A_{\epsilon,m}^{(k)}\right) + \mathbb{P}\left(E_{\hat{d},\epsilon}^{(\ell)} \mid \hat{d} \notin A_{\epsilon,m}^{(k)}\right) \mathbb{P}\left(A_{\epsilon,m}^{(k)c}\right) \\ &= \mathbb{P}\left(\bigcup_{d \in A_{\epsilon,m}^{(k)}} E_{d,\epsilon}^{(\ell)}\right) \mathbb{P}\left(A_{\epsilon,m}^{(k)}\right) + \mathbb{P}\left(E_{\hat{d},\epsilon}^{(\ell)} \mid \hat{d} \notin A_{\epsilon,m}^{(k)}\right) \mathbb{P}\left(A_{\epsilon,m}^{(k)c}\right) \\ &\leq \mathbb{P}\left(\bigcup_{d \in A_{\epsilon,m}^{(k)}} E_{d,\epsilon}^{(\ell)}\right) + \mathbb{P}\left(A_{\epsilon,m}^{(k)c}\right) \end{aligned} \quad (4.13)$$

$$\leq 2 \left|A_{\epsilon,m}^{(k)}\right| \exp\{-2\ell\epsilon^2\gamma\} + 2 \exp\left\{\frac{-2^k \rho^{(m)} \epsilon^2}{4}\right\} \quad (4.14)$$

where the last inequality follows from (4.7) and (4.12). To have the right side be no larger than $\delta > 0$ it suffices to ensure that ϵ satisfies

$$\epsilon \leq \sqrt{\frac{1}{2^k \rho^{(m)}} \ln\left(\frac{1}{\delta}\right)}$$

and

$$\epsilon \leq \sqrt{\frac{1}{2\ell\gamma} \left(\ln\left|A_{\epsilon,m}^{(k)}\right| + \ln\left(\frac{4}{\delta}\right) \right)}. \quad (4.15)$$

Now, from (4.9),

$$\left|A_{\epsilon,m}^{(k)}\right| \leq \left((1+\epsilon)2^k \rho^{(m)} \right) \frac{1}{1-\alpha_-^{-1}}.$$

Hence we may bound

$$\ln\left|A_{\epsilon,m}^{(k)}\right| \leq \ln\left((1+\epsilon)2^k \rho^{(m)} \right) + \ln\frac{1}{1-\alpha_-^{-1}}. \quad (4.16)$$

Now, the following bound on the combination number is easy to verify,

$$\binom{n}{k} \leq \frac{n^k}{k!}$$

from which we obtain

$$\begin{aligned} \ln\binom{n}{k} &\leq k \ln n - \ln \prod_{j=1}^k j \\ &= k \ln n - \sum_{j=1}^k \ln j \\ &\leq k \left(\ln\left(\frac{n}{k}\right) + 1 \right) \end{aligned}$$

where we used $\sum_{j=1}^k \ln j \geq \int_1^k \ln x \, dx$. Hence the first term of (4.16) is bounded as

$$\begin{aligned} \ln\left((1+\epsilon)2^k \rho^{(m)} \right) &\leq (1+\epsilon)\rho^{(m)}2^k \left(\ln\left(\frac{1}{(1+\epsilon)\rho^{(m)}}\right) + 1 \right) \\ &\leq 3\rho^{(m)}2^{k-1} \left(\ln\left(\frac{1}{\rho^{(m)}}\right) + 1 \right). \end{aligned}$$

Now,

$$\alpha_-^{-1} = \frac{\lceil (1+\epsilon)2^k \rho^{(m)} \rceil}{2^k - \lceil (1+\epsilon)2^k \rho^{(m)} \rceil + 1} \leq \frac{2^{k-1}}{2^k - 2^{k-1} + 1} = \frac{2^{k-1}}{2^{k-1} + 1}$$

hence

$$\frac{1}{1 - \alpha_-^{-1}} \leq 2^{k-1} + 1 \leq 2^k.$$

It follows that the second term of (4.16) is bounded from above by k . Therefore (4.16) is bounded from above by

$$3\rho^{(m)}2^{k-1} \left(\ln \left(\frac{1}{\rho^{(m)}} \right) + 1 \right) + k.$$

Hence, for any $0 < \delta \leq 1$, with confidence at least $1 - \delta$ the deviation $\left| \frac{1}{\nu} \sum_{j=1}^{\nu} \xi_j^{(\nu)} - \beta \right|$ between the frequency of 1s and β of the subsequence $\xi^{(\nu)}$ selected by the rule $\Gamma_{\hat{d}}$ based on the learnt model \mathcal{M}_k is bounded by

$$\sqrt{\max \left\{ \frac{1}{2\ell\gamma} \left(\ln \left(\frac{4}{\delta} \right) + 3\rho^{(m)}2^{k-1} \left(\ln \left(\frac{1}{\rho^{(m)}} \right) + 1 \right) + k \right), \frac{1}{2^k \rho^{(m)}} \ln \left(\frac{1}{\delta} \right) \right\}}.$$

This concludes the proof of Theorem 1.

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